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REPRESENTATION OF VISCOELASTIC
MATERIAL PROPERTIES USING
OPTIMIZATION TECHNIQUES (NASA)
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**THE ANALYTICAL REPRESENTATION OF
VISCOELASTIC MATERIAL PROPERTIES
USING OPTIMIZATION TECHNIQUES**

By S.A. Hill

Structures and Dynamics Laboratory
Science and Engineering Directorate

February 1993



National Aeronautics and
Space Administration

George C. Marshall Space Flight Center

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TECHNICAL MEMORANDUM

THE ANALYTICAL REPRESENTATION OF VISCOELASTIC MATERIAL PROPERTIES USING OPTIMIZATION TECHNIQUES

INTRODUCTION

One of the purposes of performing tests—either static, dynamic, mechanical, or electrical—is to determine the response of a system to a set of given test inputs. This test information can be reduced to a function of the system response, the dependent variable, to the test inputs, or the independent variables. This model can then be used to predict intermediate cases that are uneconomical to test. In the specific case of modeling the time-dependent properties of viscoelastic materials, it is particularly important that an accurate characterization of these properties be developed so that additional sources of error are not introduced into the solutions of problems involving these properties.

A popular method used for formulating an analytical model for these time-dependent properties is to curve fit the test data to a predefined model, such as the Prony series. The Prony series can be defined as

$$f(t) = A + \sum B_i e^{\gamma_i t}, \quad (1)$$

where t is defined as the independent variable, $f(t)$ as the dependent variable, and A , B_i , and γ_i are function constants. The difficulty in reducing test data to the Prony series model is that the number of knowns in the model is exceeded by the number of unknowns. When this is encountered using traditional methods, usually the only recourse is to assume values for some of the unknowns.

In the case of the Prony series, the values for the exponential constants, γ_i , are assumed to follow a decade pattern (i.e., $\gamma_1 = -1$, $\gamma_2 = -10$, $\gamma_3 = -100$, ...). With the γ_i terms defined, the constants A and B_i can be solved for using least-squares regression techniques. The major drawback to this method is that the function is only as accurate as the values assumed for γ_i .

Rodriguez¹ presented a method in which the Prony series could be fit to test data using two Prony terms. This method allowed for the analytical solution of all constant terms. The only drawback to this method lay in the fact that the small number of Prony terms allowed for in this method may not characterize all the subtleties of the test data, thus the need for higher order terms in the function.

The technique presented in this report allows data to be fit to the Prony series without the need for assuming values for the γ_i terms. This technique determines the "optimum" Prony series function based on the input data and on minimizing the difference between the actual and calculated dependent variable. This report will expand on the method employed to determine the "optimum" function and will present some numerical examples to demonstrate the robustness, speed, and accuracy of the method.

The computer program that was developed to perform the analysis was written in FORTRAN 77, and the analyses were performed on a 25-MHz 80386 PC with a math coprocessor installed.

DESCRIPTION OF THE TECHNIQUE

Recently, more and more emphasis has been placed on the design of structures that are light-weight yet offer high strength of stiffness. In order to achieve the minimum weight designs, optimization techniques and algorithms have been developed throughout the design community. One of the tools employed by the Structural Development Branch at Marshall Space Flight Center (MSFC) is the Design Optimization Tool (DOT) by VMA Engineering, Inc.²

DOT is a general optimization code written in FORTRAN 77 that will perform either constrained or unconstrained optimization, given the necessary design variables, constraints, and objective function. In order to access the optimization routines located within the DOT code, a header program is written that contains the definition of the initial DOT parameters; the initial value, lower bound, and upper bound for all design variables; a description of the objective function; a description of the functions required to constrain the design variables; and a calling statement to the DOT program.

There are three optimization algorithms available with the DOT code for the solution of optimization problems. The first method, the Broydon-Fletcher-Goldfarb-Shanno (BFGS) algorithm, is employed only in unconstrained optimization problems. The other two available methods, the modified method of feasible directions (MMFD) and sequential linear programming (SLP), can be employed in the solution of constrained optimization problems.

The technique developed in this report is an unconstrained optimization method and, therefore, employs the BFGS algorithm to minimize the objective function. The term "unconstrained optimization" means that the function being optimized (i.e., the objective function) is not restrained to any design space. Although this technique employs an unconstrained optimization method, it is not truly unconstrained. The design variables (i.e., variables that can be manipulated in order to achieve a minimized or maximized objective function) are restrained through the use of side constraints which limit the range of values any one design variable can assume. The relation of constraints and side constraints to the objective function are depicted in figure 1 (from ref. 2).

In the evaluation of analytical data for an unrelated problem, it was found that the correlation of a function, derived through linear least-squares techniques, to a particular set of data could be improved through slight manipulation of the constant terms of the function. This led to the implementation of the DOT code to solve for the constants of a given function through the minimization of the sum of the residuals squared. The sum of the residuals squared gives a direct measurement of the accuracy of the curve fit and is defined as

$$S_r = \sum (f(x_i) - y_i)^2, \quad (2)$$

where $f(x_i)$ is defined as the value of the derived function at a given value of the independent variable, and y_i is the value of the dependent variable at the same independent variable value. Basically, the sum of the residuals squared reveals how close the function comes to passing through all the data points. This result is then used to calculate the correlation coefficient of the function. The correlation coefficient is also used to determine the accuracy of the curve fit, but reveals the percentage of error in the fit that is not explained by the function. The correlation coefficient is given by

$$r = \sqrt{\frac{S_r - S_r}{S_t}}, \quad (3)$$

where S_t is defined as the variance of the data about its mean.

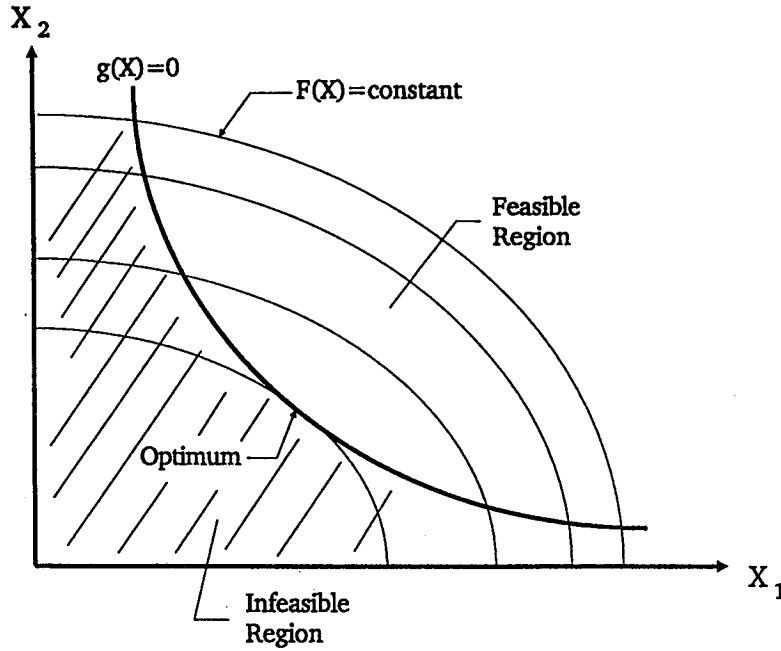


Figure 1. Relation of constraints to an objective function.

Realizing the potential for this technique in curve fitting data to a variety of functions, it was decided to evaluate the technique by using it to reduce data to a function involving the Prony series. The Prony series provides a unique opportunity to evaluate the robustness of this technique because of the large number of design variables that can be applied against a given data set. For instance, to fit data to a Prony series with seven Prony terms, the optimization problem becomes an unconstrained minimization of a highly nonlinear function containing 15 design variables. Since the Prony series has two unknowns for each Prony term, the B_i and γ_i terms, plus the constant term, A , the number of design variables can be determined by the relation

$$\text{Number of Design Variables} = 2n+1, \quad (4)$$

where n is the number of Prony terms to fit to the data.

Next, the issue of side constraints on the design variables will be discussed. In any regression of data, the constants of the function being determined have direct physical meaning to the independent and dependent variables. In the case of the Prony series being used to regress the time-dependent properties of viscoelastic materials, the A , B_i , and γ_i terms have a direct correspondence to the viscoelastic material properties.

Evaluation of the physical significance of the constants requires examination of equation (1). In order for the exponential term in the series to be dimensionless, the γ_i term will be required to have units of 1/time which is the form of a rate. Since the relaxation modulus of viscoelastic materials is a decaying function with respect to time, the γ_i term is the rate of decay of the modulus, and thus will require a side constraint to ensure that it will always be negative.

In equation (1), if time is allowed to approach infinity, then the series component of the function approaches zero and results in the constant A being the asymptotic value of the relaxation modulus at time infinity. Thus, in order to have physical significance, the constant A must always be positive. Therefore, a lower limit bound (i.e., a side constraint) of zero is set for the values that the constant A can assume.

Now, if time is allowed to approach zero, then all the exponential terms in the series component will approach one. Thus, the relaxation modulus at time zero is the summation of the constant A and the B_i terms. Therefore, the B_i terms are also directly related to the relaxation modulus and, thus, require a side constraint on the lower bound to maintain positive values for all the B_i terms.

COMPUTER IMPLEMENTATION

Appendix A contains the FORTRAN 77 source code for the program PRONY which reduces data to the Prony series form based on the implementation of the DOT optimization subroutines.

The program first initializes the variables necessary for the DOT subroutines to perform the optimization. The program then prompts the user for a data input filename, an output filename, and the number of Prony terms to be used.

Next, the side constraints for each of the design variables are set to zero and either a large positive or negative number, dependent upon whether the variable is allowed only positive or negative values, respectively. Then a call is made to a subroutine where the input file is opened, the x_i and y_i data pairs are read, and the mean value and variance of the dependent variable are calculated. Next, the loop is entered that controls the iterative process of performing the optimization task. Upon meeting the convergence criteria, the loop is exited, and the subroutine EVAL is evaluated to ensure that the objective function and the sum of the residuals squared are evaluated with the final values of the design variables. Finally, the output subroutine is called to send the results of the optimization to disk file.

Now that the subject of design variables, constraints, side constraints, and the objective function have been discussed, it is time to briefly mention some of the other parameters that are required to perform the optimization task through DOT. The first item to discuss is the convergence criteria. DOT uses two user-definable variables to control the convergence, the first being an absolute criteria and the second a relative criteria.

If the initial value of the objective function is changed by less than the value for the absolute criteria during the optimization process, then the optimization task is assumed to have converged and will terminate.² The relative criteria will stop optimization when the relative change between the objective function in any two successive iterations is less than or equal to this variable. Since the objective function generally assumes a large value on the initial calculation, because the initial values of the design variables are usually far from the optimum values, the Prony series program relies on the relative criteria to control convergence.

The other remaining item to discuss concerning the implementation of the DOT code in the reduction of data to a Prony series function is the variable that controls the number of consecutive iterations that must satisfy the convergence criteria before optimization is terminated. By default, the DOT code sets this variable equal to two, since it is common to make little progress on one iteration, only to make significant progress on the next. It is suggested that if consistent progress is made on optimizing the objective function and if computer time is inexpensive, then this value can be raised. The PRONY program automatically overrides the default value and sets the variable equal to three, based upon test runs that showed additional improvement with the increased number of iterations.

NUMERICAL EXAMPLES

Solid Rocket Motor TP-H1148 Propellant

Modeling the relaxation modulus of solid rocket motor (SRM) TP-H1148 propellant was selected as the first case since it was used as a numerical example by Rodriguez,¹ thus giving a direct comparison between the two methods. The data used by Rodriguez were obtained from a stress relaxation test of SRM TP-H1148 propellant that was tested at -30°F and 2-percent strain.³ The time varying relaxation moduli are listed in table 1.

Table 1. Stress relaxation results of SRM TP-H1148 propellant at -30°F and 2-percent strain.

Time (min)	E_r (lb/in ²)
1.66×10^{-3}	21,114
6.61×10^{-3}	12,913
1.66×10^{-2}	10,992
4.17×10^{-2}	10,114
8.32×10^{-2}	8,399
1.26×10^{-2}	7,566
1.66×10^{-1}	7,088
1.66×10^0	3,568
1.66×10^1	1,981

Rodriguez plotted the data points and fit a third-order B-spline curve to the data using an Intergraph computer-aided design (CAD) package called IGDS. Nineteen evenly spaced data points were then selected from the portion of the B-spline curve between 0.002 and 0.020 min. (Note: One of the requirements of the technique developed by Prony and used by Rodriguez is that the data points be spaced at equal time increments.) The constants A , B_1 , γ_1 , B_2 , and γ_2 were determined to be 10,973.3, 23,397.2, -380.8698 , $-79,056.7$, and $-1,971.8239$, respectively.

Comparing this curve-fitted Prony function to the actual test data through equations (2) and (3) yields a sum of residuals squared of 1.71×10^8 and a correlation coefficient of 0.566609. On initial inspection, this method does not appear to generate a precise fit of the data points, but if the function is compared to the data points that basically fall within the aforementioned 0.002- and 0.020-min time interval, then a sum of residuals squared of 1.18×10^6 and a more impressive correlation coefficient of 0.992128 are yielded.

As a comparison of the Rodriguez method and the optimization method, a series of runs was made using the Prony program and the TP-H1148 test data. These runs were made in an effort to determine the sensitivity of the method to the initial starting point of the design variables and to the number of Prony terms utilized in the analysis. The output file listing generated by the Prony program for each of the cases listed in table 2 is contained within appendix C. Each file contains the date and time of the run, the input parameters, the sum of the residuals squared, the values for the A , B_i , and γ_i terms, the correlation coefficient, and the central processing unit (CPU) time required for the analysis.

Table 2. Sensitivity study results for SRM TP-H1148 propellant.

Prony Terms	Sums of Residuals Squared	Correlation Coefficient	Run Time (s)
2	1,427,300.1	0.99716	1.4
3	82,166.3	0.99984	3.0
4	82,201.7	0.99984	3.5
5	83,250.0	0.99983	6.6

Referring to table 2, it can be seen that the first case of two Prony terms yielded a sum of residuals squared that was 99.17 percent lower than that yielded by the Rodriguez method. The remaining three cases of three, four, and five Prony terms appear to have reached a lower bound on the sum of residuals squared that is approximately 94.24 percent lower than the case with two Prony terms. In all four cases analyzed, the CPU time required to run the analyses was insignificant due to the small number of data points available for the curve fit.

Figures 2 and 3 graphically depict the relationship between the data points, the second-order Prony function derived from the Rodriguez method, and the second- and fourth-order Prony functions derived through the optimization technique, respectively. The figures show that the Rodriguez Prony function accurately models the propellant behavior over the time interval in which it was developed (0.002 to 0.020 min). It does not accurately model the late time behavior of the propellant. The second- and fourth-order optimized Prony functions, on the other hand, appear to accurately model the subtle fluctuations in the test data over the full-time interval. The apparent faceted nature of the curves in figures 2 and 3 is a direct result of the respective functions being plotted only at the test data points.

V747-75 Viton Fluoroelastomer

The second example also involves the time-dependent relaxation modulus of a viscoelastic material. This time the data were obtained from a stress relaxation test of the V747-75 Viton fluoroelastomer. The stress relaxation test was conducted by the Polymers and Composites Branch at MSFC on an RDS-7700 dynamic spectrometer at 1-percent strain⁴ and was shared with the author for the purpose of evaluating the subject technique. The data generated from this test and then used by the PRONY program are listed in appendix B.

The data were originally fit to a 14-term Prony function that was developed in the classical method of assuming values for the γ_i terms, linearizing the resulting function, and then solving for the A and B_i terms through least-squares techniques. The values for the constants A , B_i , and γ_i terms determined from this technique are given in table 3.

The resultant Prony function generated from applying these classically derived coefficients yields a sum of residuals squared of 9.20×10^6 and a correlation coefficient of 0.968443 when compared to the test data. This high correlation coefficient shows that the regressed function compares very well with the test data.

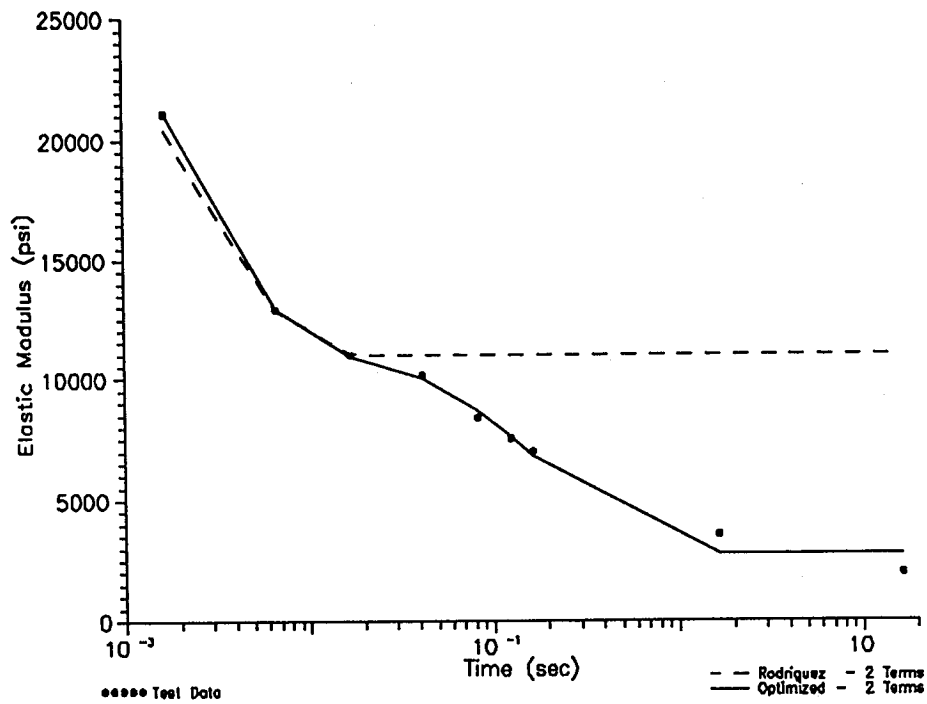


Figure 2. Comparison of Prony series functions, Rodriguez versus optimized methods.

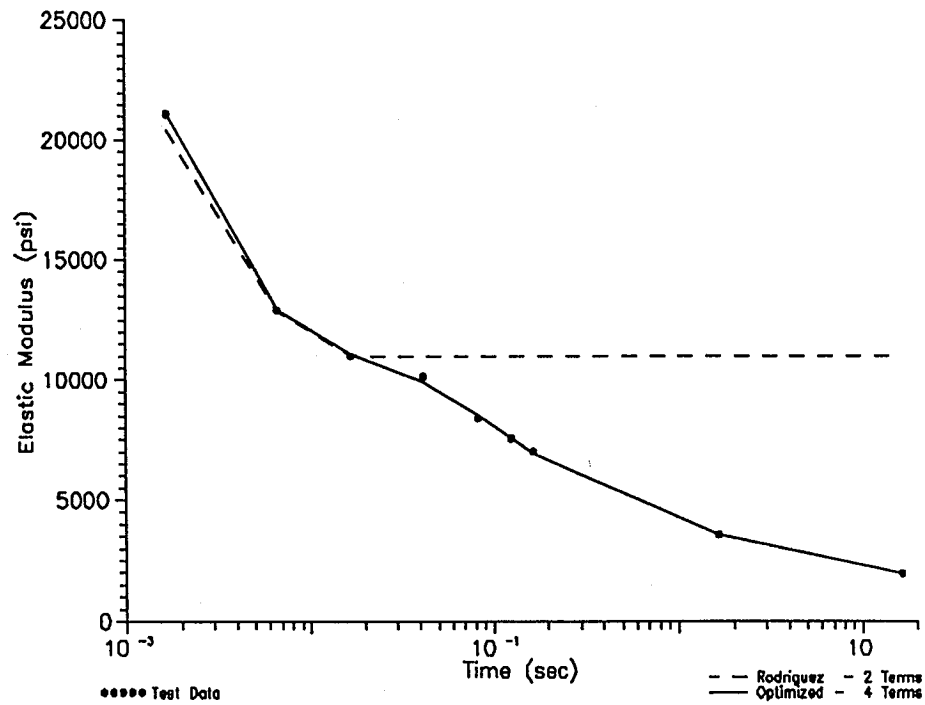


Figure 3. Comparison of Prony series functions, Rodriguez versus optimized methods.

Table 3. Classically derived Prony coefficients.

A:	111.800	γ_1 :	-1.0×10^7
B_1 :	-45,830.000	γ_2 :	-1.0×10^6
B_2 :	2,547.000	γ_3 :	-1.0×10^5
B_3 :	3,156.000	γ_4 :	-1.0×10^4
B_4 :	1,378.000	γ_5 :	-1.0×10^3
B_5 :	506.000	γ_6 :	-1.0×10^2
B_6 :	134.900	γ_7 :	-1.0×10^1
B_7 :	86.450	γ_8 :	-1.0×10^0
B_8 :	47.140	γ_9 :	-1.0×10^{-1}
B_9 :	38.420	γ_{10} :	-1.0×10^{-2}
B_{10} :	26.420	γ_{11} :	-1.0×10^{-3}
B_{11} :	24.340	γ_{12} :	-1.0×10^{-4}
B_{12} :	11.540	γ_{13} :	-1.0×10^{-5}
B_{13} :	6.586	γ_{14} :	-1.0×10^{-6}
B_{14} :	15.010		

A sensitivity study was again performed employing the PRONY program and the V747-75 Viton test data. The results of the study are contained in table 4. The output file listing generated by the PRONY program for each of the cases listed in table 4 is contained within appendix C. Each file contains the date and time of the run, the input parameters, the sum of the residuals squared, the values for the A, B_i , and γ_i terms, the correlation coefficient, and the CPU time required for the analysis.

Table 4. Sensitivity study results for V747-75 Viton.

Prony Terms	Sums of Residuals Squared	Correlation Coefficient	Run Time (s)
3	478,677.4	0.99838	34.6
4	187,225.0	0.99937	64.4
5	58,402.1	0.99980	117.4
6	32,888.1	0.99989	205.0
7	19,110.1	0.99994	355.6
8	33,641.3	0.99989	334.3
9	139,630.9	0.99953	610.6
10	19,703.0	0.99994	661.7
12	19,705.8	0.99993	842.2
14	17,484.3	0.99994	1,330.7

Referring to table 4, it can be seen that there is an indirect relationship between the number of terms and the sum of residuals squared up to and including the function containing seven Prony terms. Also, there exists a direct relationship between the number of terms and computer run time. With the exception of the nine Prony term case, it appears that after seven Prony terms, the optimized regression

has basically reached its lower limit for the sum of residuals squared. The small decreases in the sum of the residuals squared are not worth the additional run time of the program, but the CPU time on a PC is virtually free, and, if the user has the patience, the additional accuracy afforded with the 14th order Prony function may be worth the wait. An additional consideration that will affect the selection of the order of curve to use is the intended use of the function. If the function is to be used in "simple" hand calculations or short computer programs, then the higher-order curve may be selected. However, if the function is to be used in nonlinear stress analysis applications (i.e., applications in which a significant number of calculations will be performed with the function), one of the lower-order functions should be selected to reduce analysis times.

Figures 4 and 5 graphically depict the relationship between the data points, the classically derived 14th order Prony function, and the 7th and 14th order Prony functions derived through the optimization technique, respectively. These two figures reveal the excellent correlation between the data and the optimally determined Prony functions. This correlation is most pronounced at the earlier time intervals since the higher relaxation moduli at these times artificially weight the optimization method and result in a better fit. At the later times, the relaxation moduli are smaller, and, thus, differences between the function and the data at these time intervals contribute less to the sum of the residuals squared (equation (2)). Therefore, at these later times, the function has more of a tendency to "wander" from the actual data than at the earlier times.

CONCLUSIONS

Application of optimization techniques to the task of reducing experimental data to an equivalent function of the form of the Prony series has been demonstrated. This technique generates functions that show excellent correlation to their original experimental data, as shown in the previous two numerical examples. It appears from the numerical examples that the technique is fairly robust, since it reduced data sets as small as 18 data pairs and as large as 99 data pairs. Also, the technique displayed the capability to reduce data sets containing data pairs spaced at varying time increments.

It has been previously noted that this regression through optimization technique can generate functions that are artificially weighted to fit larger ordinate values more accurately than smaller ones. This artificial weighting was most apparent in the V747-75 Viton numerical example in which the function more closely approximated the experimental relaxation modulus at the early time intervals, where the modulus varied between 6,000 and 1,000 lb/in², than at later times, where the modulus was less than 1,000 lb/in². A few techniques were employed to attempt to minimize this residual weighting effect. Both normalization of the sum of residuals squared by the mean and changing the objective function so that the sum of the residuals would be minimized were attempted and generated poorer correlation of the resulting function to the experimental data. One method yet to be attempted is to decrease the frequency at which the large ordinate data pairs appear in the data set. Bower⁵ states that this technique shows promise in a similar technique.

As was stated previously, the reduction of experimental data to accurate mathematical models is a vital part of most scientific and engineering research. This technique of regression through optimization can be applied to other mathematical models that are difficult to fit to experimental data through traditional regression techniques. The author has also employed this technique to fit an n independent variable power function to experimental data with a resultant function that more accurately predicts values for the dependent variable than a function generated through traditional regression techniques. In fact, the FORTRAN code for the program PRONY that is listed in appendix A would require minimal changes to determine coefficients for a wide range of mathematical models.

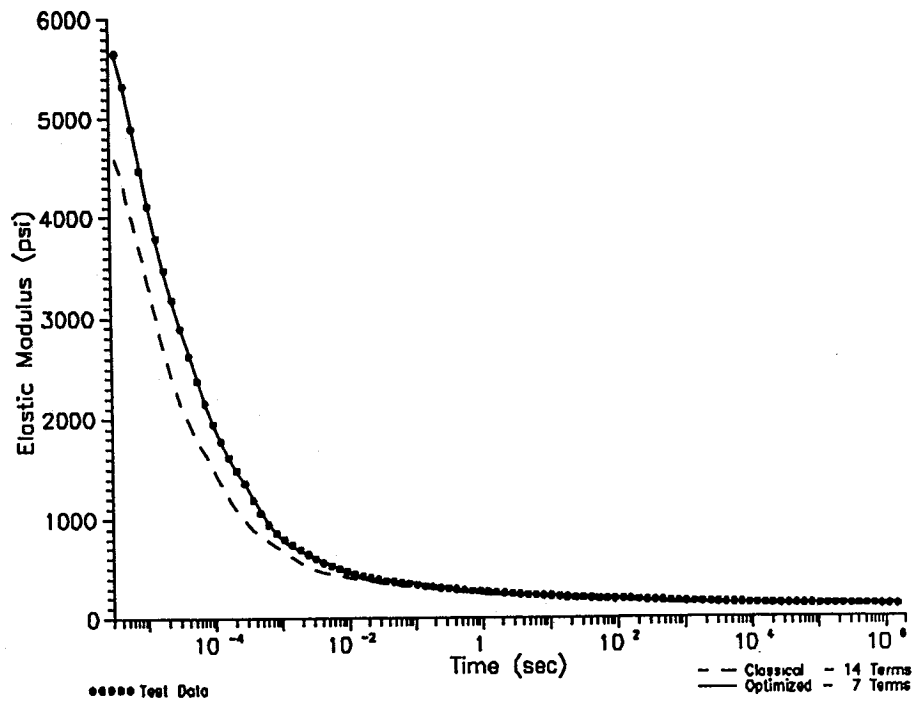


Figure 4. Comparison of Prony series functions, classical versus optimized methods.

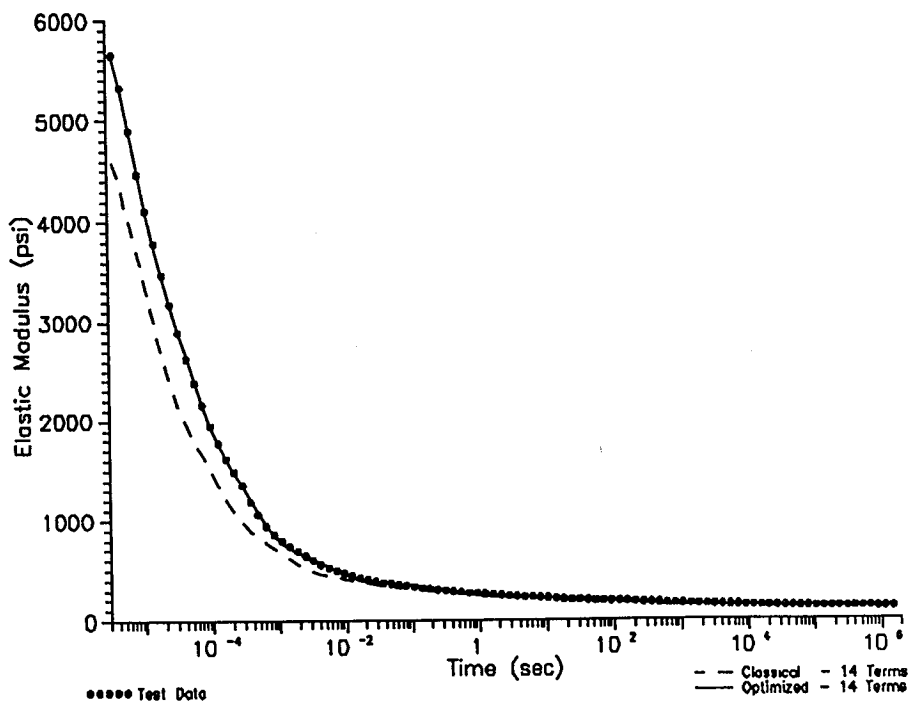


Figure 5. Comparison of Prony series functions, classical versus optimized methods.

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2. Vanderplaats, G.N.: "DOT Users Manual." Version 2.04, Vanderplaats, Miura, and Associates, Inc., 1989.
3. Harvey, A.R.: "Strain Evaluation Cylinder Study and Propellant Characterization for TP-H1148 SRM Propellant." TWR-14153, January 1984.
4. F.E. Ledbetter III, private communication.
5. Dr. M.V. Bower, private communication.

APPENDIX A

PRONY Source Code Listing

```

C ***** Prony Curve Fit *****
C * Written By: Scott A. Hill *
C *****

PARAMETER (converge=0.00001)
PARAMETER (itermax = 999)
PARAMETER (itercon = 3)

DOUBLE PRECISION r2,sumt,obj

CHARACTER*12 In,Out

DIMENSION x(41),xl(41),xu(41)
DIMENSION a(2000),y(2000)
DIMENSION wk(2200)
DIMENSION rprm(20)
DIMENSION g(1)
DIMENSION iwk(120)
DIMENSION iprm(20)

C INITIALIZE REAL AND INTEGER WORKSPACE FOR DOT

DO 10 i=1,20
  rprm(i)=0.0
  iprm(i)=0
10 CONTINUE

rprm(3) = converge
rprm(4) = converge
iprm(3) = itermax
iprm(4) = itercon

C DEFINE # OF DESIGN VARIABLES, # OF CONSTRAINTS, AMT OF
C DOT OUTPUT & WHETHER PROBLEM IS MINIMIZATION OR
C MAXIMIZATION.

nrwk=2200
nriwk=120
ncon=0
iprint=0
minmax=-1
method=1

C DEFINE INITIAL VALUES AND UPPER/LOWER BOUNDS FOR DESIGN
C VARIABLES

WRITE (*,15)
READ (*, '(A)') In
FORMAT (' Input File: '\)

WRITE (*,16)
READ (*, '(A)') Out
FORMAT (' Output File: '\)

WRITE (*,17)
READ (*,*) ndv
FORMAT (' # of Prony Terms: '\)

ndv = 2*ndv + 1
x(1) = 150.0
xl(1) = 0.0
xu(1) = 1.0E+25

DO 20 i=2,ndv,2
  x(i) = 500.0
  x(i+1) = -3000.0
20 CONTINUE

DO 25 i = 2,ndv
  xl(i) = 0.0
  xu(i) = 1.0E+25
25 CONTINUE

DO 30 i=3,ndv,2
  xl(i) = -1.0E+25
  xu(i) = 0.0
30 CONTINUE

C READ IN DATA & PERFORM STATISTICAL FUNCTIONS

CALL InData(a,y,len,sumt)

C READY TO OPTIMIZE

info=0

DO WHILE (.TRUE.)

  CALL DOT(info,method,iprint,iprint,ndv,
    ncon,x,xl,xu,obj,minmax,g,rprm,
    iprm,wk,nrwk,iwk,nriwk)

  CALL EVAL(obj,x,y,a,sumt,len,r2,ndv)

  IF (info.EQ.0) EXIT

END DO

C OPTIMIZATION COMPLETE -- OUTPUT RESULTS

CALL EVAL(obj,x,y,a,sumt,len,r2,ndv)

CALL OutPut(obj,x,ndv,r2,iprm(19),Type)

END

C*****
C SUBROUTINE TO EVALUATE FUNCTION AND CONSTRAINTS
C*****

SUBROUTINE EVAL(obj,x,y,a,sumt,len,r2,n,dv)

REAL*8 sumt,r2,Value
REAL*8 obj,fx

DIMENSION x(*)
DIMENSION y(*),a(*)

obj = 0.0

DO 100 i = 1,len
  fx = Value(x,a(i),ndv)
  obj = obj + (y(i) - fx)*(y(i) - fx)
100 CONTINUE

r2 = (sumt - obj)/sumt

RETURN
END

C*****
C FUNCTION TO CALCULATE VALUE OF PRONY SERIES
C*****

REAL*8 FUNCTION Value(array,x,n)

REAL*8 dv

DIMENSION array(*)

Value = array(1)

DO 200 i = 2,n-1,2
  dv = array(i+1)*x
  Value = Value + array(i)*DEXP(dv)
200 CONTINUE

END

```

```

C*****
C SUBROUTINE TO GET DATA FROM INPUT FILE
C*****

```

```

SUBROUTINE InData(a,y,len,sumt)

REAL*8      mean,sumt
CHARACTER*12 In

DIMENSION a(*),y(*)

io = 0
len = 0
sum = 0.0
sumt = 0.0

OPEN(1,FILE=In,STATUS='OLD')

DO 500 i = 1, 1000
  READ(1,*,END=510) a(i),y(i)
500 CONTINUE

510 CLOSE(1,STATUS='KEEP')

len = i - 1

DO 520 i = 1,len
  sum = sum + y(i)
520 CONTINUE

mean = sum/len

DO 530 i = 1,len
  sumt = sumt + (y(i) - mean)*(y(i) - mean)
530 CONTINUE

RETURN
END

```

```

C*****
C SUBROUTINE TO SEND DATA TO OUTPUT FILE
C*****

```

```

SUBROUTINE OutPut(obj,x,ndv,r2,iter)

REAL*8      obj,r2
CHARACTER*12 In,Out

DIMENSION x(*)

OPEN (2,FILE=Out,STATUS='UNKNOWN')

k = 0

WRITE (2,610) CHAR(228),CHAR(231)
WRITE (2,611) In
WRITE (2,612) 150.0,500.0,CHAR(231),-1000.0,
  CHAR(231),0.0,0.000001
WRITE(2,620) obj,x(1)

DO 600 i = 2,ndv-1,2
  k=k+1
  WRITE (2,630) k,x(i),CHAR(231),k,x(i+1)
600 CONTINUE

WRITE(2,640) DSQRT(r2),iter

610 FORMAT(/5X,'Prony Equation Form: y = A + ',
  A1,'Be^(',A1,'t)')
.
611 FORMAT(/5X,'Input File: ',A)
612 FORMAT(/5X,'Initial Value for A: ',F12.4/5X,
  'Initial Value for B: ',F12.4/5X,'Initial Value',
  ' for ',A,': ',F12.4/5X,'Upper Limit for ',A,
  ': ',F12.4 //5X,'Convergence Criteria: '
  ,F12.10)
.

```

```

613 FORMAT(/5X,'Prony Equation Form: y = A + ',A1,
  'Bxe^(',A1,'t)')
620 FORMAT(/5X,'Sum of the Residuals Squared: ',
  F14.4//',F14.4//12X,'A: ',F14.6)
630 FORMAT(10X,'B',I2,': ',F14.6,15X,A1,I2,': ',F14.6)
640 FORMAT(/5X,'Correlation Coefficient: ',F9.7//
  5X,'# of Iterations:',I3)

CLOSE (2,STATUS='KEEP')

RETURN
END

```

APPENDIX B

V747-75 Viton Stress Relaxation Data

<u>Time (min)</u>	<u>E_r (psi)</u>	<u>Time (min)</u>	<u>E_r (psi)</u>	<u>Time (min)</u>	<u>E_r (psi)</u>
3.6700E-06	5.6500E+03	2.0650E+00	2.4780E+02	1.1620E+06	1.2050E+02
4.8090E-06	5.3240E+03	2.7050E+00	2.4250E+02	1.5220E+06	1.1900E+02
6.3000E-06	4.8970E+03	3.5450E+00	2.3740E+02		
8.2550E-06	4.4720E+03	4.6450E+00	2.3130E+02		
1.0820E-05	4.1130E+03	6.0850E+00	2.2750E+02		
1.4170E-05	3.7880E+03	7.9730E+00	2.2220E+02		
1.8570E-05	3.4710E+03	1.0450E+01	2.1800E+02		
2.4330E-05	3.1760E+03	1.3690E+01	2.1390E+02		
3.1880E-05	2.8910E+03	1.7930E+01	2.0900E+02		
4.1770E-05	2.6240E+03	2.3500E+01	2.0430E+02		
5.4720E-05	2.3760E+03	3.0790E+01	2.0000E+02		
7.1700E-05	2.1530E+03	4.0340E+01	1.9600E+02		
9.3950E-05	1.9470E+03	5.2860E+01	1.9210E+02		
1.2310E-04	1.7710E+03	6.9260E+01	1.8770E+02		
1.6130E-04	1.6130E+03	9.0740E+01	1.8630E+02		
2.1130E-04	1.4810E+03	1.1890E+02	1.8490E+02		
2.7690E-04	1.3530E+03	1.5580E+02	1.8310E+02		
3.6280E-04	1.1850E+03	2.0410E+02	1.7850E+02		
4.7530E-04	1.0590E+03	2.6740E+02	1.7390E+02		
6.2280E-04	9.4020E+02	3.5040E+02	1.7080E+02		
8.1600E-04	8.5680E+02	4.5910E+02	1.6780E+02		
1.0690E-03	7.8990E+02	6.0150E+02	1.6500E+02		
1.4010E-03	7.3480E+02	7.8820E+02	1.6060E+02		
1.8350E-03	6.8780E+02	1.0330E+03	1.5820E+02		
2.4050E-03	6.4450E+02	1.3530E+03	1.5670E+02		
3.1510E-03	6.0100E+02	1.7730E+03	1.5310E+02		
4.1290E-03	5.5750E+02	2.3230E+03	1.5140E+02		
5.4090E-03	5.2570E+02	3.0440E+03	1.4950E+02		
7.0880E-03	4.9600E+02	3.9880E+03	1.4680E+02		
9.2870E-03	4.6860E+02	5.2250E+03	1.4510E+02		
1.2170E-02	4.4330E+02	6.8460E+03	1.4330E+02		
1.5940E-02	4.2210E+02	8.9700E+03	1.4160E+02		
2.0890E-02	4.0390E+02	1.1750E+04	1.3980E+02		
2.7370E-02	3.8690E+02	1.5400E+04	1.3810E+02		
3.5860E-02	3.7290E+02	2.0180E+04	1.3650E+02		
4.6990E-02	3.6500E+02	2.6440E+04	1.3600E+02		
6.1560E-02	3.5740E+02	3.4640E+04	1.3450E+02		
8.0660E-02	3.4750E+02	4.5380E+04	1.3350E+02		
1.0570E-01	3.3350E+02	5.9460E+04	1.3230E+02		
1.3850E-01	3.2100E+02	7.7910E+04	1.3130E+02		
1.8140E-01	3.1100E+02	1.0210E+05	1.2990E+02		
2.3770E-01	3.0070E+02	1.3380E+05	1.2910E+02		
3.1150E-01	2.9330E+02	1.7520E+05	1.2830E+02		
4.0810E-01	2.8600E+02	2.2960E+05	1.2770E+02		
5.3470E-01	2.7680E+02	3.0090E+05	1.2660E+02		
7.0060E-01	2.7220E+02	3.9420E+05	1.2540E+02		
9.1800E-01	2.6610E+02	5.1650E+05	1.2430E+02		
1.2030E+00	2.5990E+02	6.7670E+05	1.2320E+02		
1.5760E+00	2.5340E+02	8.8670E+05	1.2160E+02		

APPENDIX C

PRONY Program Output Files

TP-H1148 SRM Propellant

07/28/1992
10:35:48

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: srm2.dat

Initial Value for A: 2700.0000
Initial Value for B: 8000.0000
Initial Value for τ : -300.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000000100

Sum of the Residuals Squared: 1427300.1201

A: 2779.583000
B1: 8.794046E+03 τ_1 : -4.731266E+00
B2: 1.748029E+04 τ_2 : -3.604886E+02

Correlation Coefficient: .9971610

of Iterations: 36
Optimization Time: 0hr 0min 1.4sec

07/28/1992
10:49:23

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: srm2.dat

Initial Value for A: 100.0000
Initial Value for B: 100.0000
Initial Value for τ : -100.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000010000

Sum of the Residuals Squared: 82201.7363

A: 1982.994000
B1: 5.718736E+03 τ_1 : -9.948862E+00
B2: 4.273885E+03 τ_2 : -5.971107E-01
B3: 7.629745E+03 τ_3 : -3.896799E+02
B4: 1.012811E+04 τ_4 : -3.969893E+02

Correlation Coefficient: .9998367

of Iterations: 45
Optimization Time: 0hr 0min 3.5sec

07/28/1992
10:44:01

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: srm2.dat

Initial Value for A: 1000.0000
Initial Value for B: 100.0000
Initial Value for τ : -100.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000010000

Sum of the Residuals Squared: 82166.2936

A: 1980.808000
B1: 1.775723E+04 τ_1 : -3.946309E+02
B2: 4.318967E+03 τ_2 : -6.028311E-01
B3: 5.687424E+03 τ_3 : -1.008739E+01

Correlation Coefficient: .9998368

of Iterations: 54
Optimization Time: 0hr 0min 3.0sec

07/28/1992
11:09:14

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: srm2.dat

Initial Value for A: 1400.0000
Initial Value for B: 131.0000
Initial Value for τ : -100.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000010000

Sum of the Residuals Squared: 83249.9692

A: 1984.816000
B1: 3.686836E+03 τ_1 : -1.063451E+01
B2: 2.206965E+03 τ_2 : -7.734952E+00
B3: 4.170810E-01 τ_3 : -2.230149E+03
B4: 4.079599E+03 τ_4 : -5.694004E-01
B5: 1.772458E+04 τ_5 : -3.917794E+02

Correlation Coefficient: .9998346

of Iterations: 65
Optimization Time: 0hr 0min 6.6sec

V747-75 Viton Fluoroelastomer

01/22/1992
16:12:15

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -500.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 478677.4458

A:	187.385400		
B1:	3.819927E+03	$\tau 1$:	-7.623515E+04
B2:	4.861179E+02	$\tau 2$:	-2.872654E+01
B3:	1.974550E+03	$\tau 3$:	-3.894463E+03

Correlation Coefficient: .9983830

of Iterations: 93
Optimization Time: 0hr 0min 34.6sec

01/23/1992
07:49:44

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0001000000

Sum of the Residuals Squared: 58402.0883

A:	150.559100		
B1:	3.135792E+03	$\tau 1$:	-1.681769E+05
B2:	3.978183E+02	$\tau 2$:	-8.580777E+01
B3:	1.601320E+02	$\tau 3$:	-1.008694E-01
B4:	1.298270E+03	$\tau 4$:	-2.538760E+03
B5:	2.147726E+03	$\tau 5$:	-2.467912E+04

Correlation Coefficient: .9998029

of Iterations: 154
Optimization Time: 0hr 1min 57.4sec

01/22/1992
16:15:53

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -500.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 187224.9701

A:	159.108800		
B1:	2.115458E+02	$\tau 1$:	-5.073118E-01
B2:	6.906992E+02	$\tau 2$:	-3.441239E+02
B3:	2.046774E+03	$\tau 3$:	-7.607188E+03
B4:	3.635521E+03	$\tau 4$:	-9.960206E+04

Correlation Coefficient: .9993678

of Iterations: 114
Optimization Time: 0hr 1min 4.4sec

01/23/1992
07:53:54

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 32888.0913

A:	136.880500		
B1:	8.903401E+01	$\tau 1$:	-3.697306E-03
B2:	3.241039E+03	$\tau 2$:	-1.343083E+05
B3:	1.139755E+03	$\tau 3$:	-2.481861E+03
B4:	1.876921E+03	$\tau 4$:	-1.908836E+04
B5:	3.799151E+02	$\tau 5$:	-1.512457E+02
B6:	1.491300E+02	$\tau 6$:	-2.062958E+00

Correlation Coefficient: .9998890

of Iterations: 202
Optimization Time: 0hr 3min 25.0sec

01/23/1992
08:00:47

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 19110.1326

A:	137.237000	
B1:	9.063905E+01	τ_1 : -4.059385E-03
B2:	1.296831E+03	τ_2 : -2.984362E+03
B3:	1.164508E+03	τ_3 : -2.645348E+04
B4:	9.470085E+02	τ_4 : -2.577721E+04
B5:	1.540934E+02	τ_5 : -2.360951E+00
B6:	4.070435E+02	τ_6 : -1.741451E+02
B7:	3.099127E+03	τ_7 : -1.704219E+05

Correlation Coefficient: .9999355

of Iterations: 263
Optimization Time: 0hr 5min 55.6sec

01/23/1992
08:21:34

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 139630.8670

A:	139.145000	
B1:	8.233751E+01	τ_1 : -7.876419E-03
B2:	4.220577E+02	τ_2 : -1.708051E+02
B3:	1.098796E+03	τ_3 : -3.100827E+03
B4:	2.086049E+02	τ_4 : -1.831882E+04
B5:	3.555962E+03	τ_5 : -7.973099E+04
B6:	3.027768E+02	τ_6 : -1.972911E+04
B7:	1.417834E+02	τ_7 : -2.247524E+00
B8:	5.097406E+02	τ_8 : -5.760359E+03
B9:	1.100006E+01	τ_9 : -5.820372E-04

Correlation Coefficient: .9995286

of Iterations: 215
Optimization Time: 0hr 6min 50.6sec

01/23/1992
08:07:17

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 33641.3004

A:	137.107400	
B1:	1.521836E+02	τ_1 : -2.256213E+00
B2:	1.499064E+03	τ_2 : -2.030280E+04
B3:	9.005947E+01	τ_3 : -3.925191E-03
B4:	3.887046E+02	τ_4 : -1.622622E+02
B5:	3.163455E+01	τ_5 : -1.420035E+04
B6:	1.140985E+03	τ_6 : -2.566299E+03
B7:	3.298198E+02	τ_7 : -1.568566E+04
B8:	3.214954E+03	τ_8 : -1.324574E+05

Correlation Coefficient: .9998864

of Iterations: 207
Optimization Time: 0hr 5min 34.3sec

01/23/1992
08:33:42

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 19073.0373

A:	132.907500	
B1:	6.299915E+01	τ_1 : -9.576948E-04
B2:	8.757581E+01	τ_2 : -1.771211E-01
B3:	5.747751E+02	τ_3 : -2.629937E+03
B4:	1.420293E+02	τ_4 : -9.763288E+00
B5:	6.023077E+02	τ_5 : -1.751084E+04
B6:	3.735197E+02	τ_6 : -2.204888E+02
B7:	5.726660E+02	τ_7 : -2.890676E+03
B8:	1.243688E+03	τ_8 : -2.652360E+04
B9:	3.099371E+03	τ_9 : -1.429671E+05
B10:	1.601951E+02	τ_{10} : -9.859158E+03

Correlation Coefficient: .9999356

of Iterations: 289
Optimization Time: 0hr 11min 1.7sec

01/23/1992
11:34:44

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -1000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 19705.8298

A:	133.527800		
B1:	1.482939E+02	$\tau 1$:	-1.012032E+01
B2:	8.787807E+01	$\tau 2$:	-1.609448E-01
B3:	6.217831E+01	$\tau 3$:	-8.913591E-04
B4:	3.951779E+02	$\tau 4$:	-2.483659E+02
B5:	2.622941E+02	$\tau 5$:	-6.500080E+03
B6:	1.728592E+03	$\tau 6$:	-2.389782E+04
B7:	4.072714E+01	$\tau 7$:	-3.761625E+03
B8:	9.704579E+01	$\tau 8$:	-9.570082E+03
B9:	9.301317E+02	$\tau 9$:	-2.696142E+03
B10:	3.821681E+01	$\tau 10$:	-3.602862E+03
B11:	1.437028E-03	$\tau 11$:	-2.693170E+04
B12:	3.131319E+03	$\tau 12$:	-1.417388E+05

Correlation Coefficient: .9999335

of Iterations: 263
Optimization Time: 0hr 14min 2.2sec

01/23/1992
12:23:32

Prony Equation Form: $y = A + \sum [Be^{\tau t}]$

Input File: data.rlx

Initial Value for A: 150.0000
Initial Value for B: 500.0000
Initial Value for τ : -3000.0000
Upper Limit for τ : .0000

Convergence Criteria: .0000100000

Sum of the Residuals Squared: 17484.2568

A:	136.978600		
B1:	7.829514E+01	$\tau 1$:	-6.173314E+03
B2:	1.521367E+02	$\tau 2$:	-2.203388E+00
B3:	3.104636E+03	$\tau 3$:	-1.868573E+05
B4:	1.371974E-01	$\tau 4$:	-1.328096E+04
B5:	6.143808E+01	$\tau 5$:	-7.706898E+03
B6:	3.986375E+02	$\tau 6$:	-1.654526E+02
B7:	2.540747E+01	$\tau 7$:	-4.293449E+03
B8:	1.882670E-03	$\tau 8$:	-5.806276E+03
B9:	8.968185E+01	$\tau 9$:	-3.824855E-03
B10:	8.744885E+00	$\tau 10$:	-1.269737E+04
B11:	1.186864E+03	$\tau 11$:	-2.782989E+03
B12:	9.729550E+02	$\tau 12$:	-2.831181E+04
B13:	1.141888E+03	$\tau 13$:	-2.859916E+04
B14:	7.512561E+01	$\tau 14$:	-1.803128E+04

Correlation Coefficient: .9999410

of Iterations: 322
Optimization Time: 0hr 22min 10.7sec

APPROVAL

THE ANALYTICAL REPRESENTATION OF VISCOELASTIC MATERIAL PROPERTIES USING OPTIMIZATION TECHNIQUES

By S.A. Hill

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.



J.C. BLAIR

Director, Structures and Dynamics Laboratory

